MATHEMATICS

B.Sc, Part-II, Paper III

Infinite Series

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THEOREM D'Alembert's Ratio Test

Prove that the series $\sum u_n$ of positive terms is convergent if $\frac{u_n}{u_{n-1}} < K < 1$, where K is a fixed number and n has any value. The series is divergent if $\frac{u_n}{u_{n-1}} \geq$ for all n.

Proof (i) Let the series beginning from the fixed term be

$$u_1 + u_2 + u_3 + u_4 + \cdots \text{ to } \infty,$$

and let $\frac{u_2}{u_1} < K, \frac{u_3}{u_2} < K, \frac{u_4}{u_3} < K \cdots$, where $K < 1$.

Then $u_1 + u_2 + u_3 + u_4 + \cdots$ to ∞

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \cdots \text{ to } \infty\right)$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \cdots \text{ to } \infty\right)$$

i.e., $u_1 + u_2 + u_3 + \cdots$ to $\infty < u_1(1 + K + K^2 + K^3 + \cdots \text{ to } \infty)$

i.e., $u_1 + u_2 + u_2 + \cdots$ to $\infty < \frac{u_1}{1-K}$, as $K < 1$.

Hence the given series is convergent.

(ii) Let $S_n = u_1 + u_2 + u_3 + u_4 + \cdots + u_n$,

where $\frac{u_2}{u_1} \geq 1, \frac{u_3}{u_2} \geq 1, \frac{u_4}{u_3} \geq 1, \cdots$

i.e. $u_2 \geq u_1, u_3 \geq u_2 \geq u_1, u_4 \geq u_3 \geq u_2 \geq u_1 \cdots$

$\therefore S_n \geq u_1 + u_1 + u_1 + u_1 + \cdots$ to n terms

i.e., $S_n \geq n u_1$.

Taking limits of both sides, we get

$$\lim_{n\to\infty} S_n \geq \lim_{n\to\infty} n u_1$$

i.e. $S_n \to \infty$ as $n \to \infty$

Hence the series is divergent.

EXAMPLES

$$1 + \frac{2^p}{\lfloor 2} + \frac{3^p}{\lfloor 3} + \frac{4^p}{\lfloor 4} + \cdots \text{ to } \infty, \text{ for all values of } p.$$

Solution Let the n^{th} term of the given series be denoted by u_n.

Then $u_n = \frac{n^p}{\lfloor n}$

Replacing n by $n+1$, we get $u_{n+1} = \frac{n+1}{\lfloor n+1}$.

$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)}{(n+1)\lfloor n} \cdot \frac{\lfloor n}{n^p} = \left(1+\frac{1}{n}\right)^p \cdot \frac{1}{1+n};$

or $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p \cdot \lim_{n\to\infty} \frac{1}{1+n} = 0 < 1,$ for all values of p.

Hence, by D'Alembert's ratio test, the given series is convergent for all values of p.

(2) Discuss the convergence of the series

$$\frac{\lfloor 1}{1^1} + \frac{\lfloor 2}{2^2} + \frac{\lfloor 3}{3^3} + \cdots \text{ to } \infty$$

or $\sum_1^\infty \frac{\lfloor n}{n^n}$.

Solution. Let the n^{th} term of the given series be denoted by u_n

Then $u_n = \frac{\lfloor n}{n^n}$. $\therefore u_{n+1} = \frac{\lfloor n+1}{(n+1)^{n+1}}$.

$\therefore \frac{u_{n+1}}{u_n} = \frac{\lfloor n+1}{(n+1)^{n+1}} \cdot \frac{n^n}{\lfloor n} = \frac{(n+1)\lfloor n}{(n+1)(n+1)^n} \cdot \frac{n^n}{\lfloor n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$.

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$, as $2 < e < 3$.

Hence, by D'Alembert's ratio test, the given series is convergent.

(3) Test the convergence of the series

$$\frac{1}{4} + \frac{1\cdot 3}{4\cdot 7} + \frac{1\cdot 3\cdot 5}{4\cdot 7\cdot 10} + \frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 7\cdot 10\cdot 13} + \cdots \text{ to } \infty$$

Solution. Let the n^{th} term of the given series be denoted by u_n.

The n^{th} term of $1, 3, 5, 7, \ldots$ is $a + (n-1)d$
$= 1 + (n-1)\cdot 2 = 2n-1$.

The n^{th} term of $4, 7, 10, 13, \ldots$ is $a + (n-1)d$
$= 4 + (n-1)\cdot 3 = 3n+1$.

$\therefore u_n = \frac{1\cdot 3\cdot 5\cdot 7 \cdots (2n-1)}{4\cdot 7\cdot 10\cdot 13 \cdots (3n+1)}$.

Replacing n by $(n+1)$, we get

$u_{n+1} = \frac{1\cdot 3\cdot 5\cdot 7 \cdots (2n-1)(2n+1)}{4\cdot 7\cdot 10\cdot 13 \cdots (3n+1)(3n+4)}$.

$\therefore \frac{u_{n+1}}{u_n} = \frac{2n+1}{3n+4} = \frac{2+\frac{1}{n}}{3+\frac{4}{n}}$

or $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{3+\frac{4}{n}} = \frac{2}{3} < 1$.

Hence by D'Alembert's ratio test, the given series is convergent.

(4) Determine the convergence of the series

$$2x+\frac{3x}{8}+\frac{4x^3}{27}+\cdots+\frac{n+1}{n^3}x^n+\cdots \text{ to } \infty, \ (x>0)$$

<u>Solution</u> Let the given series by $\sum u_n$.

Then $u_n=\frac{n+1}{n^3}x^n$.

Replacing n by $(n+1)$, we get $u_{n+1}=\frac{n+2}{(n+1)^3}x^{n+1}$.

$$\therefore \frac{u_{n+1}}{u_n}=\frac{1}{(1+\frac{1}{n})^3}\,\frac{1+\frac{2}{n}}{1+\frac{1}{n}}x. \quad \text{or } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=x.$$

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as $x<1$ or >1.

when $x=1$, then this test fails

Now $u_n=\frac{n+1}{n^3}$.

Let us consider as auxiliary series $\sum v_n$ whose n^{th} term

$v_n=\frac{1}{n^2}$.

Then $\frac{u_n}{v_n}=\frac{n+1}{n}=1+\frac{1}{n}$

or $\lim_{n\to\infty}\frac{u_n}{v_n}=1$, which is finite and non-zero.

So by comparison test, $\sum u_n$ and $\sum v_n$ will behave alike

Now $v_n=\frac{1}{n^2}$.

$$\therefore \sum v_n=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots \text{ to } \infty.$$

5) Test for convergence the series

$$\frac{2}{1}x+\frac{7}{15}x^2+\frac{12}{53}x^3+\cdots+\frac{5n-3}{2n^3-1}x^n+\cdots \text{ to } \infty \ (x>0).$$

<u>Solution</u> Here $u_n=\frac{5n-3}{2n^3-1}x^n$

and $$u_{n+1}=\frac{5(n+1)-3}{2(n+1)^3-1}x^{n+1}=\frac{5n+2}{2n^3+6n^2+6n+1}x^{n+1}.$$

$$\therefore \frac{u_{n+1}}{u_n}=\frac{5n+2}{2n^3+6n^2+6n+1}\cdot\frac{2n^3-1}{5n-3}\cdot\frac{x^{n+1}}{x^n}$$

$$=\frac{5+\frac{2}{n}}{5-\frac{3}{n}}\cdot\frac{2-\frac{1}{n^3}}{2+\frac{6}{n}+\frac{6}{n^2}+\frac{1}{n^3}}\cdot x.$$

$$\therefore \operatorname*{Lt}_{n\to\infty}\frac{u_{n+1}}{u_n}=\frac{5}{5}\cdot\frac{2}{2}\cdot x=x.$$

By ratio test, the given series is convergent if $x<1$ and divergent if $x>1$.

when $x=1$, $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = 1$ and the ratio test fails.

when $x=1$, $u_n = \frac{5n-3}{2n^3-1}$. Let $v_n = \frac{1}{n^2}$

Then $\frac{u_n}{v_n} = \frac{5n^3-3n^2}{2n^3-1} = \frac{5-\frac{3}{n}}{2-\frac{1}{n^3}}$

$\therefore \underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \frac{5}{2}$, which is finite and non-zero.

By comparison test, $\sum u_n$ converges as $\sum v_n$ converges.

6) Test the convergence of the series whose n^{th} term is $\frac{x^n}{2n^2+3}$, $(x>0)$

Solution Here $u_n = \frac{x^n}{2n^2+3}$ and $u_{n+1} = \frac{x^{n+1}}{2(n+1)^2+3}$.

$\therefore \frac{u_{n+1}}{u_n} = \frac{2n^2+3}{2(n+1)^2+3} \cdot \frac{x^{n+1}}{x^n} = \frac{2+\frac{3}{n^2}}{2(1+\frac{1}{n})^2+\frac{3}{n^2}} \cdot x$;

$\therefore \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \frac{2}{2} \cdot x = x$.

$\therefore$ By ratio test, $\sum u_n$ is convergent when $x<1$ and divergent when $x>1$.

when $x=1$, $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = 1$ and the ratio test fails.

when $x=1$, $u_n = \frac{1}{2n^2+3}$

Let $v_n = \frac{1}{n^2}$.

$\therefore \frac{u_n}{v_n} = \frac{n^2}{2n^2+3} = \frac{1}{2+\frac{3}{n^2}}$

$\therefore \underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \frac{1}{2}$, which is finite and non-zero.

$\therefore$ By comparison test $\sum u_n$ and $\sum v_n$ converge or diverge together.

But v_n i.e. $\sum \frac{1}{n^2}$ converges. Therefore $\sum u_n$ also converges, when $x=1$.

Thus the given series is convergent when $x \le 1$ and divergent when $x>1$.